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# Random Walk Weakly Attracted to a Wall

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**Abstract** We consider a random walk $X_n$ in $\mathbb{Z}_+$, starting at $X_0 = x \geq 0$, with transition probabilities

$$\mathbb{P}(X_{n+1} = X_n \pm 1 | X_n = y \geq 1) = \frac{1}{2} \mp \frac{\delta}{4y + 2\delta}$$

and $X_{n+1} = 1$ whenever $X_n = 0$. We prove $\mathbb{E}X_n \sim \text{const.}\, n^{1-\frac{\delta}{2}}$ as $n \nearrow \infty$ when $\delta \in (1, 2)$. The proof is based upon the Karlin-McGregor spectral representation, which is made explicit for this random walk.

**Keywords** Random walk · Orthogonal polynomials · Pinning · Wetting

## 1 Introduction

Random walks have been used in many different fields of physics, economics, biology... Usually, it evolves in a translation invariant environment or a random environment whose average is translation invariant. Random walks in inhomogeneous environment, in particular with a reflecting or attracting or repelling wall, have been used to mimic the behaviour of a liquid interface on top of a solid substrate, *wetting* phenomena, and a variety of other phenomena associated with the *pinning* of an interface. This is the motivation for the model

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presented below, whose specific form was chosen so as to derive rigorously and explicitly the result of competition between a relatively long ranged attraction and reflection at the wall.

We consider a random walk $X_n$ in $\mathbb{Z}_+$, defined by $X_0 = x \geq 0$,

$$
\begin{aligned}
p_y &= \mathbb{P}(X_{n+1} = X_n + 1 | X_n = y \geq 1) = \frac{1}{2} - \frac{\delta}{4y + 2\delta} = \frac{y}{2y+\delta}, \\
q_y &= \mathbb{P}(X_{n+1} = X_n - 1 | X_n = y \geq 1) = \frac{1}{2} + \frac{\delta}{4y + 2\delta} = \frac{y+\delta}{2y+\delta}
\end{aligned}
\tag{1.1}
$$

and $X_{n+1} = 1$ whenever $X_n = 0$, i.e. $p_0 = 1$, $q_0 = 0$. Let $\mathbb{P}_x(\cdot)$ denote the corresponding probabilities, and $\mathbb{E}_x(\cdot)$ the corresponding expectation values.

The walk obeys the detailed balance condition with respect to the measure $\pi$ on $\mathbb{Z}_+$ defined up to a multiplicative constant by

$$
\pi_y = \pi_0 \prod_{z=0}^{y-1} \frac{p_z}{q_{z+1}} = \pi_0 \frac{(2y+\delta)\Gamma(\delta+1)\Gamma(y)}{\Gamma(y+\delta+1)}, \quad y \geq 1 \tag{1.2}
$$

which obeys $\pi_y \sim y^{-\delta}$ when $y \to \infty$. We restrict our attention to $\delta > 1$, and normalise $\pi$ as a probability measure, with $\pi_0 = (\delta - 1)/(2\delta)$. The dynamics is periodic, with the odd and even components exchanged under one step of the dynamics, $\pi = (\pi_{\text{even}} + \pi_{\text{odd}})/2$, with $\pi_{\text{even}} = (2\pi_0, 0, 2\pi_2, \ldots)$, and $\pi_{\text{odd}} = (0, 2\pi_1, 0, 2\pi_3, \ldots)$. Starting from $X_0 = 0$, we have convergence in law $X_{2n} \to X_\infty^{\text{even}}$ and $X_{2n+1} \to X_\infty^{\text{odd}}$. The first moment exists only for $\delta > 2$, with $\mathbb{E}X_\infty = \mathbb{E}(X_\infty^{\text{even}} + X_\infty^{\text{odd}})/2 = \delta/(2(\delta - 2))$.

We focus our attention to $1 < \delta < 2$, so that $\mathbb{E}_x X_n \to \infty$ as $n \nearrow \infty$. For $|z| < 1$ let

$$
g_e(z) = \sum_{n \text{ even}} z^n \mathbb{E}_0 X_n, \quad g_o(z) = \sum_{n \text{ odd}} z^n \mathbb{E}_0 X_n \tag{1.3}
$$

Our main result is the following theorem:

**Theorem 1** *Let $\delta \in (1, 2)$. Then as $z \nearrow 1$ or $n \nearrow \infty$,*

$$
g_e(z) = \Gamma\left(2 - \frac{\delta}{2}\right) K_\delta \, (1-z)^{\frac{\delta}{2}-2}\left(1 + \mathcal{O}\left((1-z)^{(1-\frac{\delta}{2})(\delta-1)}\right)\right), \tag{1.4}
$$

$$
\mathbb{E}_0 X_n = K_\delta \, n^{1-\frac{\delta}{2}}\left(1 + \mathcal{O}\left(n^{-(1-\frac{\delta}{2})(\delta-1)}\right)\right) \tag{1.5}
$$

*where $K_\delta$ is a constant depending upon $\delta$ as defined in (4.12).*

*Remark 1* The constant $K_\delta \nearrow +\infty$ when $\delta \searrow 1$ or $\delta \nearrow 2$, but the $\mathcal{O}(\cdot)$ are not uniform in $\delta$, the theorem says nothing about $\delta = 1$ or $\delta = 2$.

*Remark 2* The odd generating function $g_o(z)$ obeys the same bounds as $g_e(z)$. For any starting point $x$, $\mathbb{E}_x X_n$ obeys the same bounds as $\mathbb{E}_0 X_n$.

*Remark 3* (1.5) follows from (1.4) by a Tauberian theorem [3, Thm. 5, p. 447]. This uses monotonicity of $\mathbb{E}_0 X_n$ for $n$ even, which holds for any reflected random walk, as recalled below together with more specific monotonicity arguments.

Consider two walks: $X_n$ started at $x$, with parameter $\delta$, and $X'_n$ started at $x'$, with parameter $\delta'$. Assume $\delta \geq \delta'$ and $x \leq x'$ and $x' - x$ even. The two walks may be coupled, e.g. using a single random number for both walks when they meet. This implies that $\mathbb{E}_x X_n$ is for each $n$ an increasing function of $x$ over the even integers, and also over the odd integers, and a decreasing function of $\delta \in (-1, +\infty)$. Monotonicity in $\delta$ is an example of a general monotonicity property in the transition probabilities $p_y, q_y$. Then

$$\mathbb{E}_0 X_{n+2} = \mathbb{P}_0(X_2 = 0)\mathbb{E}_0 X_n + \mathbb{P}_0(X_2 = 2)\mathbb{E}_2 X_n \geq \mathbb{E}_0 X_n \tag{1.6}$$

Therefore $\mathbb{E}_0 X_n$ is an increasing function of $n$ over the even integers, and also over the odd integers. And the same property holds for $\mathbb{E}_1 X_n$. But $\mathbb{E}_x X_n$ for $x \geq 2$ is not monotonous in $n$, even or odd. We have

$$\mathbb{E}(X_{n+1} - X_n | X_n = y) = -\frac{\delta}{2y + \delta}, \quad y \geq 1 \tag{1.7}$$

so that $\mathbb{E}_0 X_{2n} < \mathbb{E}_0 X_{2n-1}$. And since $\mathbb{E}_0 X_n$ is increasing over the even (or the odd) integers, we have $\mathbb{E}_0 X_{2n+1} > \mathbb{E}_0 X_{2n}$.

Equation (1.4) will be proven in Sect. 4. In Sect. 2 we exhibit the orthogonality measure of our random walk polynomials: this is Theorem 2, our main technical result, opening the way for Theorem 1. In Sect. 3 we solve a differential equation for generating functions, differential equation associated with the recursion formula for random walk polynomials.

Our random walk is of a very special form, but it could be used for comparison or as input for more realistic models. For example the polymer pinning model of Alexander and Zygouras [1] combines i.i.d. disorder with a spatially inhomogeneous Markov chain, which could be built from our random walk.

Other aspects of this model and closely related models will be treated in a forthcoming paper [4].

## 2 Random Walk Polynomials and their Orthogonality Measure

Our first tool is the Karlin-McGregor representation theorem [5]: let $L^2(\pi)$ denote the Hilbert space of complex sequences $(f_y)_{y \in \mathbb{Z}_+}$ obeying $\sum_{y=0}^{\infty} |f_y|^2 \pi_y < \infty$. Then

$$(Tf)_y = p_y f_{y+1} + q_y f_{y-1} \tag{2.1}$$

defines in $L^2(\pi)$ a self-adjoint operator $T$ of norm less or equal to one. Let $e_0 = (1, 0, 0, \ldots)$. The Karlin-McGregor representation theorem gives

$$\mathbb{P}_x(X_n = y) = \frac{\pi_y}{\pi_0} \langle T^n Q_x(T) Q_y(T) e_0, e_0 \rangle \tag{2.2}$$

where $\{Q_y(t)\}_{y \in \mathbb{Z}_+}$ is a family of polynomials of degree $y$ in $t$, defined recursively by $Q_{-1} = 0$, $Q_0 = 1$, and

$$t Q_y = p_y Q_{y+1} + q_y Q_{y-1}, \quad y \geq 0 \tag{2.3}$$

giving polynomials $Q_y$ of degree $y$ and parity $(-1)^y$, with $Q_y(1) = 1$ for all $y \geq 0$. Using the spectral resolution $\{E_t\}$ of the self-adjoint operator $T$, and $d\mu(t) = d\langle E_t e_0, e_0 \rangle$, one gets

$$\mathbb{P}_x(X_n = y) = \frac{\pi_y}{\pi_0} \int_{-1}^{1} d\mu(t) t^n Q_x(t) Q_y(t) \tag{2.4}$$

which implies that $\{Q_y(t)\}_{y\in\mathbb{Z}_+}$ is a family of orthogonal polynomials in the probability measure $d\mu(t)$,

$$\int_{-1}^{1} d\mu(t) Q_x(t) Q_y(t) = \frac{\pi_0}{\pi_y}\delta_{x,y}, \quad x, y \ge 0 \tag{2.5}$$

Given the family $\{Q_y(t)\}_{y\in\mathbb{Z}_+}$, (2.5) characterizes a unique probability measure $d\mu$, termed the orthogonality measure of the family. Letting $n \nearrow \infty$ in (2.4) shows that [5, pp. 70–71]

$$d\mu(t) = \pi_0\Big(\delta(t-1) + \delta(t+1)\Big) + d\mu^c(t) \tag{2.6}$$

where $d\mu^c(t)$ is absolutely continuous with respect to the Lebesgue measure. Indeed as $n \nearrow \infty$ with $n + y - x$ even, the LHS of (2.4) tends to $2\pi_y$, while the contribution from $d\mu^c$ to the RHS tends to zero.

Our second tool is Dette's theorem [2]: the orthogonality measure $d\mu^1$ of the first associated polynomials of a random walk is related to the orthogonality measure $d\mu^*$ of the dual random walk through

$$d\mu^1(t) = \frac{1}{q_1}(1-t^2) d\mu^*(t) \tag{2.7}$$

The first associated polynomials $Q^1_y$ are defined by $Q^1_{-1} = 0$, $Q^1_0 = 1$ and

$$t Q^1_y = p_{y+1} Q^1_{y+1} + q_{y+1} Q^1_{y-1}, \quad y \ge 0 \tag{2.8}$$

The dual random walk polynomials $Q^*_y$ are defined by $p^*_y = q_y$ and $q^*_y = p_y$ for all $y$ except $p^*_0 = 1$ and $q^*_0 = 0$, so that $Q^*_0 = 1$, $Q^*_1 = t$ and

$$t Q^*_y = p^*_y Q^*_{y+1} + q^*_y Q^*_{y-1} = q_y Q^*_{y+1} + p_y Q^*_{y-1}, \quad y \ge 1 \tag{2.9}$$

Dette's theorem may be applied starting from the dual, giving

$$d\mu^{*,1}(t) = \frac{1}{p_1}(1-t^2) d\mu(t) \tag{2.10}$$

The first associated dual polynomials $Q^{*,1}_y$ are defined by $Q^{*,1}_{-1} = 0$, $Q^{*,1}_0 = 1$ and

$$t Q^{*,1}_y = q_{y+1} Q^{*,1}_{y+1} + p_{y+1} Q^{*,1}_{y-1}, \quad y \ge 0 \tag{2.11}$$

The general definitions (2.3), (2.8), (2.9), (2.11) are made explicit by inserting our $p_y$'s and $q_y$'s, giving

$$\begin{aligned} (2y+\delta) t Q_y &= y Q_{y+1} + (y+\delta) Q_{y-1}, \quad y \ge 1,\ Q_1(t) = t, \\ (2y+2+\delta) t Q^1_y &= (y+1) Q^1_{y+1} + (y+1+\delta) Q^1_{y-1}, \quad y \ge 0, \\ (2y+\delta) t Q^*_y &= (y+\delta) Q^*_{y+1} + y Q^*_{y-1}, \quad y \ge 1, \\ (2y+2+\delta) t Q^{*,1}_y &= (y+1+\delta) Q^{*,1}_{y+1} + (y+1) Q^{*,1}_{y-1}, \quad y \ge 0 \end{aligned} \tag{2.12}$$

We thus have four distinct families of orthogonal polynomials, and the corresponding four distinct orthogonality measures $\mu$, $\mu^1$, $\mu^*$, $\mu^{*,1}$.

The third and last step is to relate our polynomials to Gegenbauer polynomials of index $\lambda$, defined by $G^{\lambda}_{-1}=0$, $G^{\lambda}_0=1$ and

$$(2y+2\lambda)tG^{\lambda}_y=(y+1)G^{\lambda}_{y+1}+(y-1+2\lambda)G^{\lambda}_{y-1},\quad y\geq 0 \tag{2.13}$$

It appears that $Q^1_y=G_y^{\frac{\delta}{2}+1}$, so that $Q^{1,*}_y=G_y^{\frac{\delta}{2}+1,*}$, and also $Q^*_y=\frac{y!\Gamma(\delta)}{\Gamma(y+\delta)}G_y^{\frac{\delta}{2}}$. These in turn satisfy

$$G_y^{\frac{\delta}{2}+1,*}(t)=\frac{(y+1)!\Gamma(\delta+1)}{\Gamma(y+1+\delta)}G_y^{\frac{\delta}{2},1}(t)=\frac{(y+1)!\Gamma(\frac{\delta+3}{2})}{\Gamma(y+\frac{\delta+3}{2})}P_y^{\frac{\delta-1}{2},\frac{\delta-1}{2}}(t;1) \tag{2.14}$$

where $P^{\alpha,\beta}_x(t;c)$ are $c$-associated Jacobi polynomials, or Wimp polynomials [6]. The orthogonality measure $d\mu^{*,1}$ is therefore also the orthogonality measure of the $P_y^{\frac{\delta-1}{2},\frac{\delta-1}{2}}(t;1)$ polynomials, namely [6, Th. 3, p. 996]

$$d\mu^{*,1}(t)=\frac{(1-t^2)^{\frac{\delta-1}{2}}}{|F(t)|^2}dt\Big/\text{normalisation} \tag{2.15}$$

where

$$\begin{aligned}F(t)&={}_2F_1\left(1,1-\delta;\frac{3-\delta}{2};\frac{1+t}{2}\right)\\&\quad+Ke^{i\pi\frac{\delta-1}{2}}\left(\frac{1+t}{2}\right)^{\frac{\delta-1}{2}}{}_2F_1\left(\frac{1+\delta}{2},\frac{1-\delta}{2};\frac{1+\delta}{2};\frac{1+t}{2}\right)\\&={}_2F_1\left(1,1-\delta;\frac{3-\delta}{2};\frac{1+t}{2}\right)+Ke^{i\pi\frac{\delta-1}{2}}\left(\frac{1-t^2}{4}\right)^{\frac{\delta-1}{2}}\\&=-1+\mathcal{O}\big((1-t)^{\frac{\delta-1}{2}}\big)\quad\text{as}\quad t\to 1\end{aligned} \tag{2.16}$$

with ${}_2F_1$ the Gauss hypergeometric function and

$$K=\frac{\Gamma(\delta)\Gamma(\frac{1-\delta}{2})}{\Gamma(\frac{\delta-1}{2})} \tag{2.17}$$

We have $F(-t)=-F(t)^*$. Then (2.6), (2.10) and (2.15) give

$$d\mu^c(t)=\frac{(1-t^2)^{\frac{\delta-3}{2}}}{|F(t)|^2}dt\Big/\text{normalisation} \tag{2.18}$$

which yields:

**Theorem 2** *The orthogonality measure of the family of polynomials defined by* (2.3) *with* (1.1) *and* $1<\delta<2$ *is the even probability measure on* $[-1,1]$ *defined by*

$$d\mu(t)=\pi_0\Big(\delta(t-1)+\delta(t+1)\Big)\;+\;d\mu^c(t) \tag{2.19}$$

*where* $\delta(\cdot)$ *is the Dirac measure at* $0$ *and* $d\mu^c$ *is given by* (2.16)–(2.18) *and* $\pi_0=\frac{\delta-1}{2\delta}$, *with the normalisation* $\int_{-1}^1 d\mu^c=\delta^{-1}$.

## 3 Generating Function

Using (2.4) and $X_n \le X_0 + n$, we have

$$\mathbb{E}_0 X_n = \sum_{y=1}^{n} \frac{y\pi_y}{\pi_0} \int_{-1}^{1} d\mu(t) t^n Q_y(t) \tag{3.1}$$

For $n$ even, using $Q_y$ orthogonal to $Q_0 \equiv 1$, and for $y$ odd $Q_y$ also orthogonal to $t^n$, and then using (2.6), we have

$$\begin{aligned}\mathbb{E}_0 X_n &= \sum_{\substack{y=2\\ \text{even}}}^{n} \frac{y\pi_y}{\pi_0} \int_{-1}^{1} d\mu^c(t)(t^n - 1) Q_y(t) \\ &= 2\int_0^1 d\mu^c(t)(t^n - 1) \sum_{\substack{y=0\\ \text{even}}}^{n} \frac{y\pi_y}{\pi_0} Q_y(t)\end{aligned} \tag{3.2}$$

The corresponding generating function is defined as

$$\begin{aligned}g_e(z) &= \sum_{n\,\text{even}} z^n \mathbb{E}_0 X_n \\ &= 2\int_0^1 d\mu^c(t) \sum_{n\,\text{even}} z^n (t^n - 1) \sum_{\substack{y=0\\ \text{even}}}^{n} \frac{y\pi_y}{\pi_0} Q_y(t) \\ &= 2\int_0^1 d\mu^c(t) \sum_{y\,\text{even}} \frac{y\pi_y}{\pi_0} Q_y(t) \left( \frac{(zt)^y}{1 - z^2 t^2} - \frac{z^y}{1 - z^2} \right)\end{aligned} \tag{3.3}$$

For $n$ odd, using $Q_y$ orthogonal to $Q_1 \equiv t$ for $y \ge 2$, and (2.5) for $y = 1$,

$$\begin{aligned}\mathbb{E}_0 X_n &= \sum_{y=1}^{n} \frac{y\pi_y}{\pi_0} \int_{-1}^{1} d\mu(t)(t^n - t) Q_y(t) + \frac{\pi_1}{\pi_0} \int_{-1}^{1} d\mu(t) t Q_1(t) \\ &= 1 + 2\int_0^1 d\mu^c(t)(t^n - t) \sum_{\substack{y=1\\ \text{odd}}}^{n} \frac{y\pi_y}{\pi_0} Q_y(t) = \mathbb{E}_1 X_{n-1}\end{aligned} \tag{3.4}$$

and the corresponding generating function

$$\begin{aligned}g_o(z) &= \sum_{n\,\text{odd}} z^n \mathbb{E}_0 X_n \\ &= \frac{z}{1 - z^2} + 2\int_0^1 d\mu^c(t) \sum_{n\,\text{odd}} z^n (t^n - t) \sum_{\substack{y=1\\ \text{odd}}}^{n} \frac{y\pi_y}{\pi_0} Q_y(t) \\ &= \frac{z}{1 - z^2} + 2\int_0^1 d\mu^c(t) \sum_{y\,\text{odd}} \frac{y\pi_y}{\pi_0} Q_y(t) \left( \frac{(zt)^y}{1 - z^2 t^2} - \frac{t z^y}{1 - z^2} \right)\end{aligned} \tag{3.5}$$

The recursion (2.3) (2.12) defining the random walk polynomials gives a poor uniform bound for these polynomials, e.g. $|Q_y(t)| < 3^y$ for $t \in (-1, 1)$. We define, for $|u| < 1/3$,

$$\Psi_t(u) = \sum_{y=1}^{\infty} \frac{\pi_y}{\pi_0} Q_y(t) u^y = \Gamma(\delta+1) \sum_{y=1}^{\infty} (2y+\delta) \frac{\Gamma(y)}{\Gamma(y+\delta+1)} Q_y(t) u^y \tag{3.6}$$

and aim at an analytic continuation giving, for $|z| < 1$,

$$\begin{aligned} g_e(z) &= z \int_0^1 d\mu^c(t) \left[ \left( \frac{t\Psi_t'(zt)}{1-z^2t^2} - \frac{\Psi_t'(z)}{1-z^2} \right) - \left( \frac{t\Psi_t'(-zt)}{1-z^2t^2} - \frac{\Psi_t'(-z)}{1-z^2} \right) \right], \\ g_o(z) &= \frac{z}{1-z^2} + z \int_0^1 d\mu^c(t) t \left[ \left( \frac{\Psi_t'(zt)}{1-z^2t^2} - \frac{\Psi_t'(z)}{1-z^2} \right) + \left( \frac{\Psi_t'(-zt)}{1-z^2t^2} - \frac{\Psi_t'(-z)}{1-z^2} \right) \right] \end{aligned} \tag{3.7}$$

The recursion (2.3) (2.12) may be converted into a first order differential equation for $\Psi_t(u)$. We first get a differential equation for

$$\Phi_t(u) = \Gamma(\delta+1) \sum_{y=1}^{\infty} \frac{\Gamma(y)}{\Gamma(y+\delta+1)} Q_y(t) u^y \equiv \sum_{y=1}^{\infty} H_y(t) u^y \tag{3.8}$$

with

$$H_y(t) = \frac{1}{2y+\delta} \frac{\pi_y}{\pi_0} Q_y(t) = \frac{\Gamma(\delta+1)\Gamma(y)}{\Gamma(y+\delta+1)} Q_y(t), \quad y \ge 1 \tag{3.9}$$

obeying, with any arbitrary value for $H_0$,

$$(2y+\delta) t H_y = (y+\delta+1) H_{y+1} + (y-1) H_{y-1}, \quad y \ge 2 \tag{3.10}$$

with

$$H_1(t) = \frac{t}{\delta+1}, \qquad H_2(t) = \frac{Q_2(t)}{(\delta+1)(\delta+2)} = \frac{(\delta+2)t^2 - (\delta+1)}{(\delta+1)(\delta+2)} \tag{3.11}$$

Multiplying (3.10) by $u^y$ and summing over $y \ge 2$ yields

$$(1 - 2tu + u^2)\Phi_t'(u) = t - u - \delta(u^{-1} - t)\Phi_t(u) \tag{3.12}$$

with $\Phi_t(0) = 0$, whose solution is

$$\Phi_t(u) = \delta^{-1} - u^{-\delta}(1 - 2tu + u^2)^{\frac{\delta}{2}} \int_0^u dv\, v^{\delta-1}(1 - 2tv + v^2)^{-\frac{\delta}{2}} \tag{3.13}$$

We thus get:

**Lemma 3** *The function $\psi_t(u)$ defined in (3.6) may be expressed as*

$$\Psi_t(u) = 2u\Phi_t'(u) + \delta\Phi_t(u) = -1 + \frac{\delta(1-u^2)}{1-2tu+u^2} \left(\delta^{-1} - \Phi_t(u)\right) \tag{3.14}$$

*where $\Phi_t(u)$ is the solution of the differential equation (3.12). It extends to an analytic function in the disc $|u| < 1$. Its derivative may be expressed as*

$$\Psi_t'(u) = \frac{\delta(1-u^2)}{u\,(1-2tu+u^2)} - \frac{B_t(u)\,\delta}{u^\delta\,(1-2tu+u^2)^{2-\frac{\delta}{2}}} \int_0^u \frac{dv\, v^{\delta-1}}{(1-2tv+v^2)^{\frac{\delta}{2}}} \tag{3.15}$$

*with*

$$B_t(u) = 4u(1-t) - 2t(1-u)^2 + \frac{\delta(1-ut)(1-u^2)}{u} \tag{3.16}$$

## 4 Proof of (1.4)

The leading order and next to leading order in (3.7) as $z \nearrow 1$ are found in

$$g_1(z) = \int_0^1 d\mu_c(t)\left(\frac{zt\Psi_t'(zt)}{1-z^2t^2} - \frac{z\Psi_t'(z)}{1-z^2}\right) \tag{4.1}$$

Indeed the leading orders come from the singularity at $t=1$ in the integral. In (3.7) there is a symmetry or anti-symmetry as $z \to -z$ and $t \to -t$ jointly, associated with the even/odd symmetries. The terms not included in (4.1) correspond to $-z \simeq -1$, not singular with $t > 0$. For $1 - z \ll 1$ and $1 - t \ll 1$ we have

$$\begin{aligned} \frac{t}{1-z^2t^2} - \frac{1}{1-z^2} &= -(1-t)\frac{1+z^2t^2}{(1-z^2t^2)(1-z^2)} \sim -\frac{1}{2}\frac{1-t}{1-z}\frac{1}{(1-z)+(1-t)}, \\ \frac{t}{1-z^2t^2} + \frac{1}{1-z^2} &= \frac{t(1-z^2)+1-z^2t^2}{(1-z^2t^2)(1-z^2)} \sim \frac{1}{2}\frac{1}{1-z}\frac{2(1-z)+1-t}{(1-z)+(1-t)} \end{aligned} \tag{4.2}$$

(3.15) (3.16) may be written as

$$q_t(z)^2\Psi_t'(z) = \delta\frac{1-z^2}{z}q_t(z) - B_t(z)\delta z^{-\delta}q_t(z)^{\frac{\delta}{2}}\int_0^z dv\, v^{\delta-1}q_t(v)^{-\frac{\delta}{2}} \tag{4.3}$$

with

$$q_t(z) = 1 - 2tz + z^2 = (1-z)^2 + 2z(1-t), \qquad q_t'(z) = 2(z-t) \tag{4.4}$$

Together with (4.2), we have to estimate $\Psi_t'(zt) \pm \Psi_t'(z)$ as $z \nearrow 1$, $t \nearrow 1$. We have

$$\frac{1-z^2t^2}{zt} \sim \frac{1-z^2}{z}\left(1 + \frac{1-t}{1-z}\right), \tag{4.5}$$

$$\begin{aligned} B_t(z) &\sim 4(1-t) - 2(2-\delta)(1-z)(1-t) + 2(\delta-1)(1-z)^2, \\ B_t'(z) &\sim 2(2-\delta)(1-t) - 4(\delta-1)(1-z), \\ B_t(zt) &\sim B_t(z)\left(1 - (1-t)\frac{B_t'(z)}{B_t(z)}\right), \end{aligned} \tag{4.6}$$

$$\begin{aligned} \int_0^1 \frac{dv\, v^{\delta-1}}{q_t(v)^{\frac{\delta}{2}}} &= \int_0^1 \frac{dv}{((1-v)^2+2(1-t))^{\frac{\delta}{2}}} + \int_0^1 dv\left[\frac{v^{\delta-1}}{q_t(v)^{\frac{\delta}{2}}} - \frac{1}{((1-v)^2+2(1-t))^{\frac{\delta}{2}}}\right] \\ &= (1-t)^{\frac{1-\delta}{2}}\int_0^{(1-t)^{-\frac{1}{2}}} \frac{dx}{(x^2+2)^{\frac{\delta}{2}}} + \mathcal{O}(1) \\ &= (1-t)^{\frac{1-\delta}{2}}\frac{2^{-\frac{1+\delta}{2}}\sqrt{\pi}\,\Gamma(\frac{\delta-1}{2})}{\Gamma(\frac{\delta}{2})} + \mathcal{O}(1) \end{aligned} \tag{4.7}$$

The range (0, 1) of the integral in (4.1) is split into four intervals, according to

$$0 < 1-(1-z)^\alpha < 1-(1-z)^\beta < 1-(1-z)^\gamma < 1 \tag{4.8}$$

with $0<\alpha<1<\beta<2<\gamma$ and respective contributions denoted $g_1^\alpha, g_1^{\alpha\beta}, g_1^{\beta\gamma}, g_1^\gamma$. We begin with the leading contribution, $g_1^{\alpha\beta}$:

- $1-(1-z)^\alpha < t < 1-(1-z)^\beta$:
  $q_t(z)\sim 2(1-t)$ and $B_t(z)\sim 4(1-t)$. Then (4.3) with (4.7) yields

$$(1-t)^2\Psi_t'(z)\sim \delta(1-z)(1-t)-(1-t)^{\frac{3}{2}}\delta\sqrt{\frac{\pi}{2}}\frac{\Gamma(\frac{\delta-1}{2})}{\Gamma(\frac{\delta}{2})} \tag{4.9}$$

  and

$$\begin{aligned}
&\int_{1-(1-z)^\alpha}^{1-(1-z)^\beta} dt\,(1-t^2)^{\frac{\delta-3}{2}}\left(\frac{t}{1-z^2t^2}-\frac{1}{1-z^2}\right)\Psi_t'(z)\\
&\sim 2^{\frac{\delta-5}{2}}\delta\sqrt{\frac{\pi}{2}}\frac{\Gamma(\frac{\delta-1}{2})}{\Gamma(\frac{\delta}{2})}\int_{1-(1-z)^\alpha}^{1-(1-z)^\beta} dt\,(1-t)^{\frac{\delta}{2}-2}\frac{1-t}{1-z}\frac{1}{(1-z)+(1-t)}\\
&=2^{\frac{\delta-5}{2}}\delta\sqrt{\frac{\pi}{2}}\frac{\Gamma(\frac{\delta-1}{2})}{\Gamma(\frac{\delta}{2})}(1-z)^{\frac{\delta}{2}-2}\int_{(1-z)^{\beta-1}}^{(1-z)^{\alpha-1}} dx\,\frac{x^{\frac{\delta}{2}-1}}{1+x}\\
&=2^{\frac{\delta-5}{2}}\delta\sqrt{\frac{\pi}{2}}\Gamma\left(\frac{\delta-1}{2}\right)\Gamma\left(1-\frac{\delta}{2}\right)(1-z)^{\frac{\delta}{2}-2}\\
&\quad\times\left(1+\mathcal{O}\big((1-z)^{(1-\alpha)(1-\frac{\delta}{2})}+(1-z)^{(\beta-1)\frac{\delta}{2}}\big)\right)
\end{aligned} \tag{4.10}$$

  Collecting all previous error terms we get

$$\begin{aligned}
g_1^{\alpha\beta}(z)&=\Gamma\left(2-\frac{\delta}{2}\right)K_\delta(1-z)^{\frac{\delta}{2}-2}\big[1+\mathcal{O}\big((1-z)^{(1-\alpha)(1-\frac{\delta}{2})}+(1-z)^{\alpha\frac{\delta-1}{2}}+(1-z)^{\frac{\alpha}{2}}\\
&\quad+(1-z)^{(\beta-1)\frac{\delta}{2}}+(1-z)^{2-\beta}\big)\big]
\end{aligned} \tag{4.11}$$

  with

$$K_\delta=2^{\frac{\delta-5}{2}}\sqrt{\frac{\pi}{2}}\frac{\Gamma(\frac{\delta-1}{2})}{1-\frac{\delta}{2}}\Big/\int_{-1}^{1}\frac{(1-t^2)^{\frac{\delta-3}{2}}}{|F(t)|^2}dt \tag{4.12}$$

  and $F(t)$ given by (2.16). The five error terms in $\mathcal{O}(\cdots)$ in (4.11) come respectively from:

  1) Extending the $x$-integral to $+\infty$ in (4.10)
  2) $F(t)=-1+\mathcal{O}((1-t)^{\frac{\delta-1}{2}})$
  3) $\Psi_t'(zt)-\Psi_t'(z)\sim\delta$, derived from (4.3)–(4.6), with the leading contribution from (4.5)
  4) Extending the $x$-integral to 0 in (4.10)
  5) $q_t(z)\sim 2(1-t)(1+\frac{(1-z)^2}{2(1-t)})$ and $B_t(z)\sim 4(1-t)(1+(\delta-1)\frac{(1-z)^2}{2(1-t)})$.

  Choosing

$$\alpha=\sup_{\delta\in(1,2)}\left(\min\left\{(1-\alpha)\left(1-\frac{\delta}{2}\right),\alpha\frac{\delta-1}{2}\right\}\right)=2-\delta \tag{4.13}$$

and $\beta$ within $4-2\sqrt{2}<\beta<2$, e.g. $\beta=3/2$, then brings (4.11) to (1.4). The remaining ranges in the $t$-integral also fall within the previous error estimates:

- $0<t<1-(1-z)^{\alpha}$:

$$g_1^{\alpha}=(1-z)^{\frac{\delta}{2}-2}\mathcal{O}\big((1-z)^{(1-\alpha)(1-\frac{\delta}{2})}+(1-z)^{\frac{\alpha}{2}}+(1-z)^{\alpha\frac{\delta-1}{2}}\big) \tag{4.14}$$

- $1-(1-z)^{\beta}<t<1-(1-z)^{\gamma}$:

$$g_1^{\beta\gamma}=(1-z)^{\frac{\delta}{2}-2}\mathcal{O}\big((1-z)^{(\beta-1)\frac{\delta}{2}}+(1-z)^{2-\beta}\big) \tag{4.15}$$

- $1-(1-z)^{\gamma}<t<1$:

$$g_1^{\gamma}=\mathcal{O}\big((1-z)^{-1}\big) \tag{4.16}$$

This completes the proof of (1.4) and Theorem 1.

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